CSE 590K: Analysis and Control of Computing Systems Using Linear Discrete-Time System Theory:

Basics of LTI Systems

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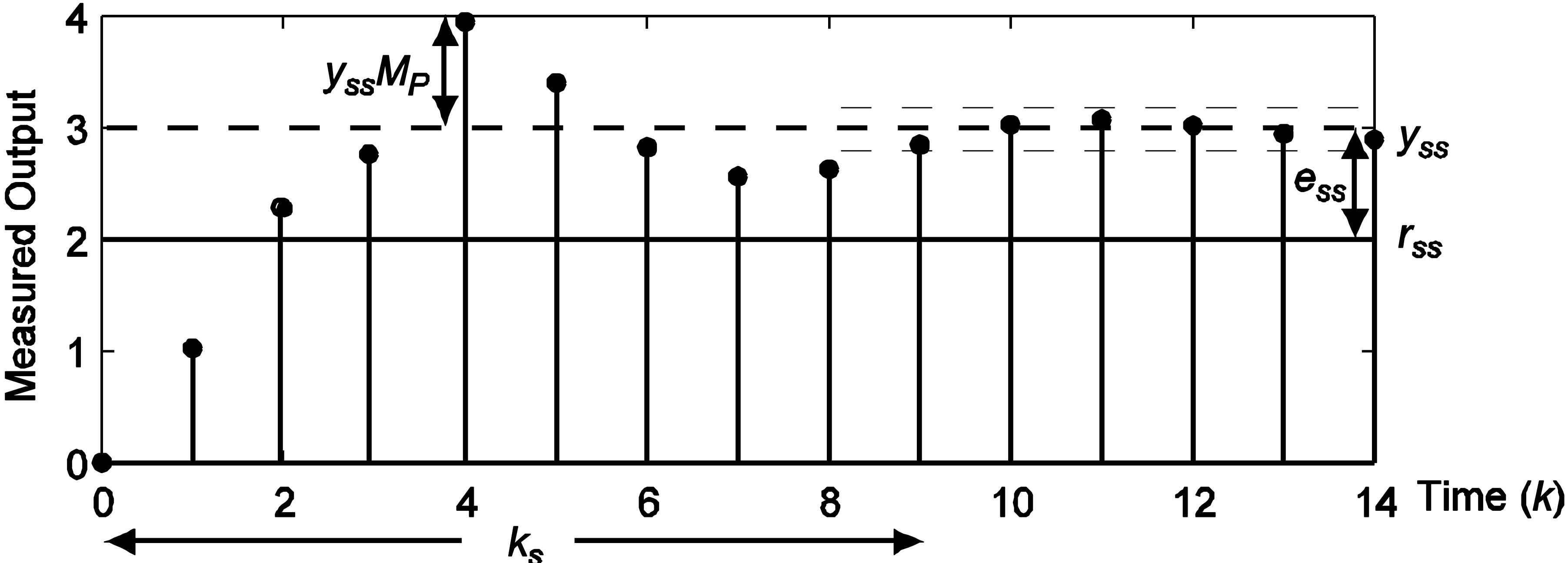
Remember SASO

- Stability
- Accuracy
- Settling time
- Overshoot

Today's Objective:
Learn theory required to analyze SASO properties of LTI Systems.

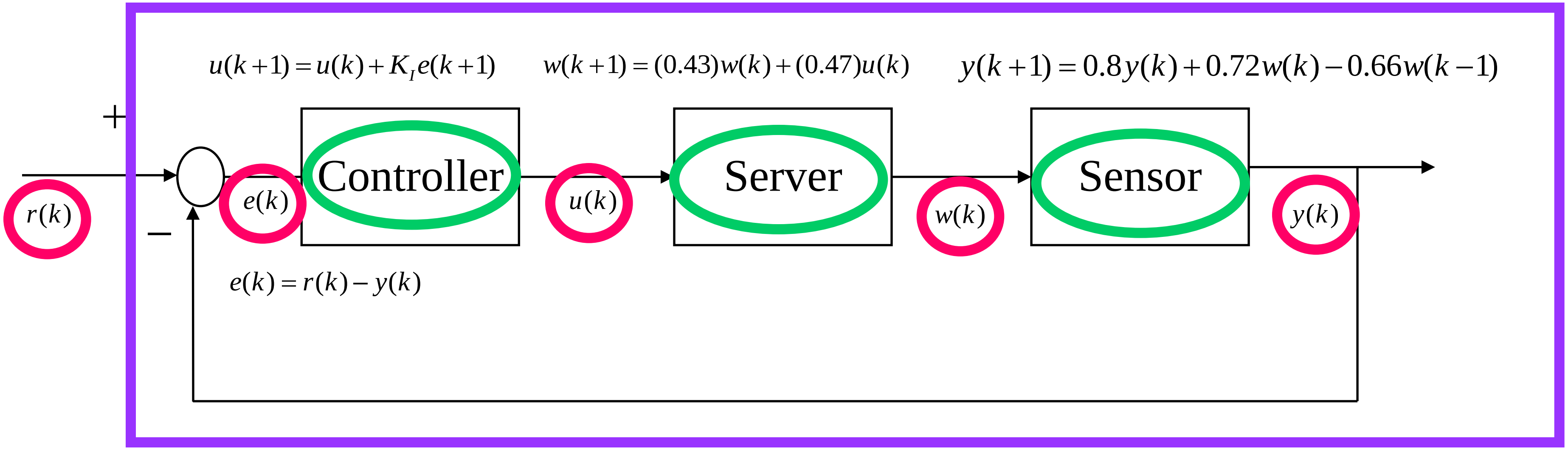


Agenda

- Signals & Z-Transforms
- Systems & Transfer Functions
- Lab

Reference: "Feedback Control of Computing Systems": Chapter 3

Motivating Example



The problem

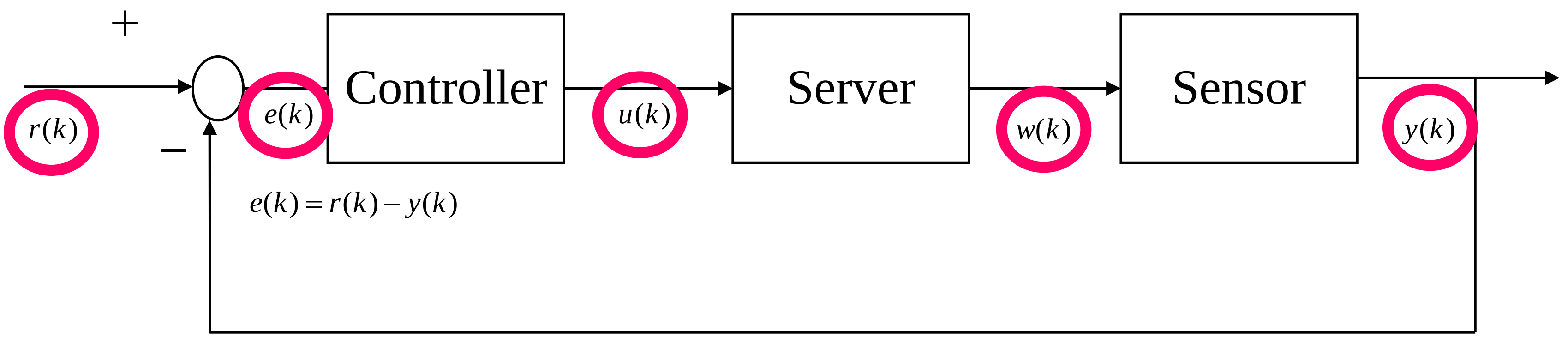
Want to find *y(k)* in terms of K_I so can design control system that is stable, accurate, settles quickly, and has small overshoot.

a. Signals

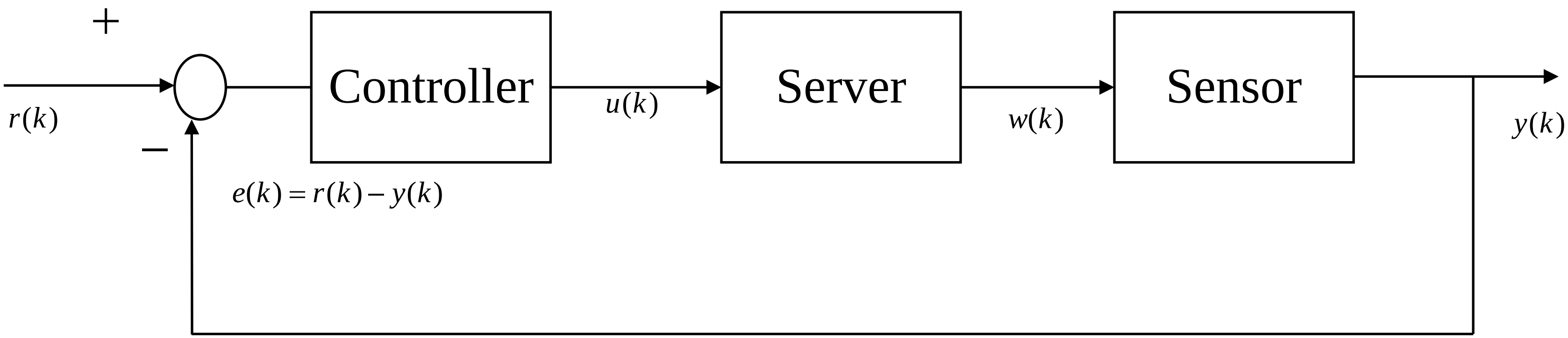
b. Transfer functions

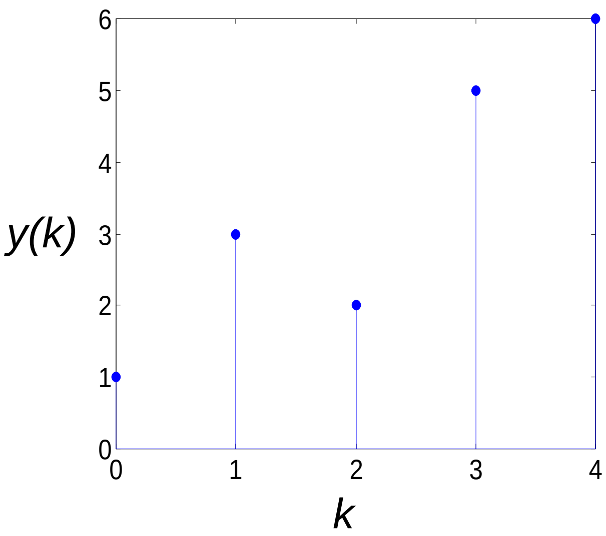
c. Composition of components – end-to-end system

Part a: Signals & Z-Transforms



Signals





Time domain representation

y(0)=1

y(1)=3

y(2)=2

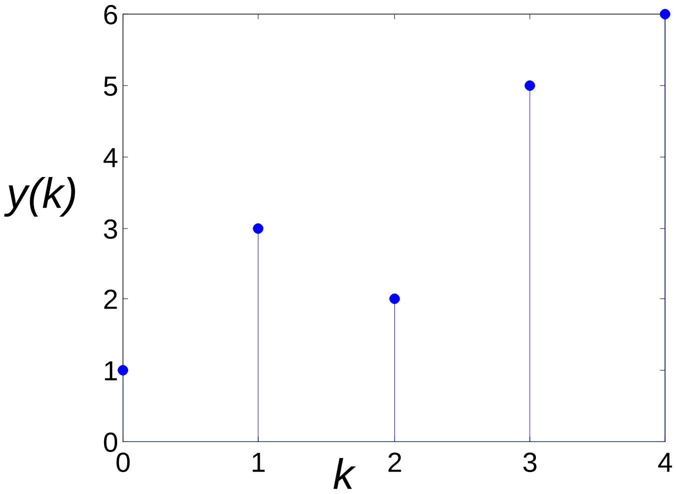
y(3)=5

y(4)=6

A signal is a real-valued function of time.

Issue: Time domain analysis is cumbersome in studying complicated control systems

Z-Transform of a Signal



Time domain representation	***z* domain representation**
$y(0)=1$	$1z^{0}$ +
$y(1)=3$	$3z^{-1}$ +
$y(2)=2$	$2z^{-2}$ +
$y(3)=5$	$5z^{-3}$ +
$y(4)=6$	$6z^{-4}$

z is time shift;
z^{-1} is time delay

$z^{0}=1$: $k=0$ (current time)

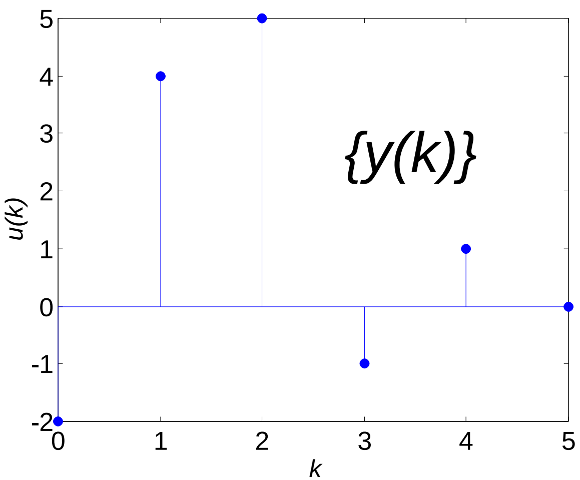
z^{-1}: $k=1$ (one time unit in the future)

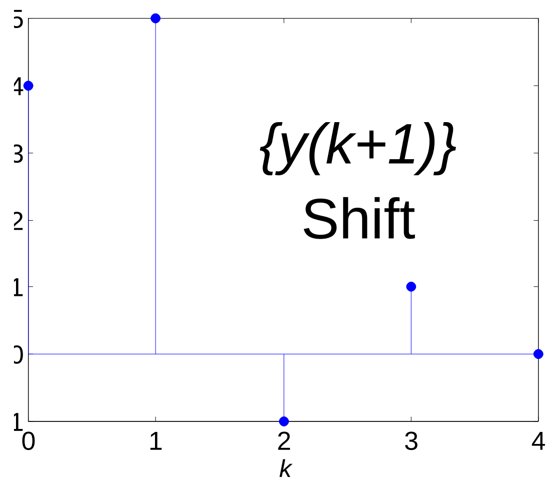
z^{-2}: $k=2$ (two time units in the future)

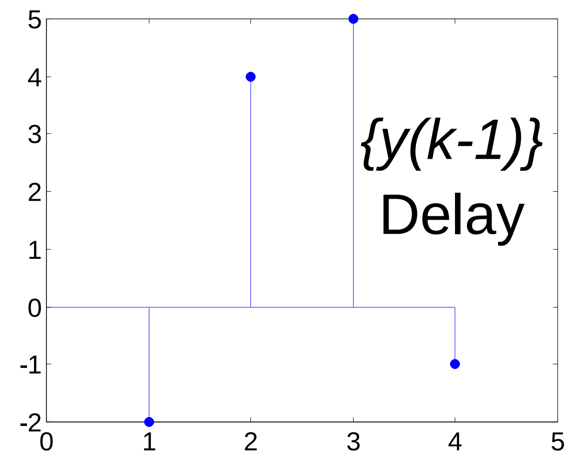
If $\{y(k)\}=y(0), y(1), \ldots$ is a signal, then its z-Transform is $Y(z)=\sum_{k=0}^{\infty} y(k) z^{-k}$

Signal Shifts and Delays

$$V(z) = zU(z) = 4 + 5z^{-1} - z^{-2} + z^{-3}$$







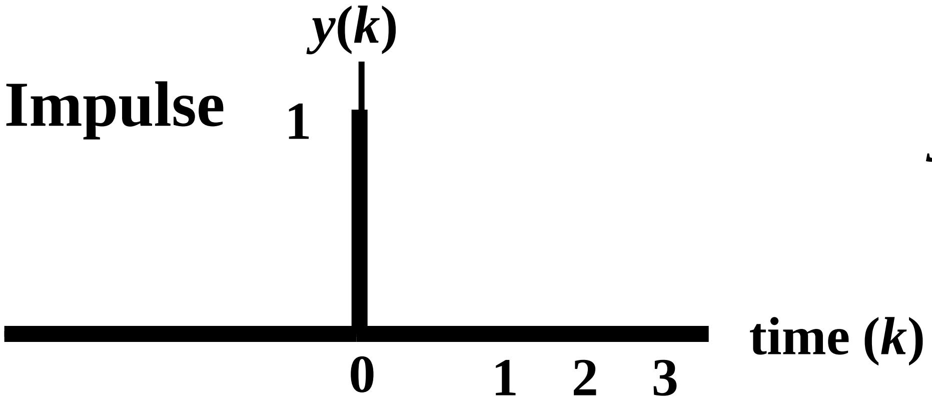
$$U(z) = -2 + 4z^{-1} + 5z^{-2} - z^{-3} + z^{-4}$$

(Drop exponents >0.)

$$V(z) = z^{-1}U(z) = 2z^{-1} + 4z^{-2} + 5z^{-3} - 1z^{-1}$$

Common Signals: Impulse

Impulse



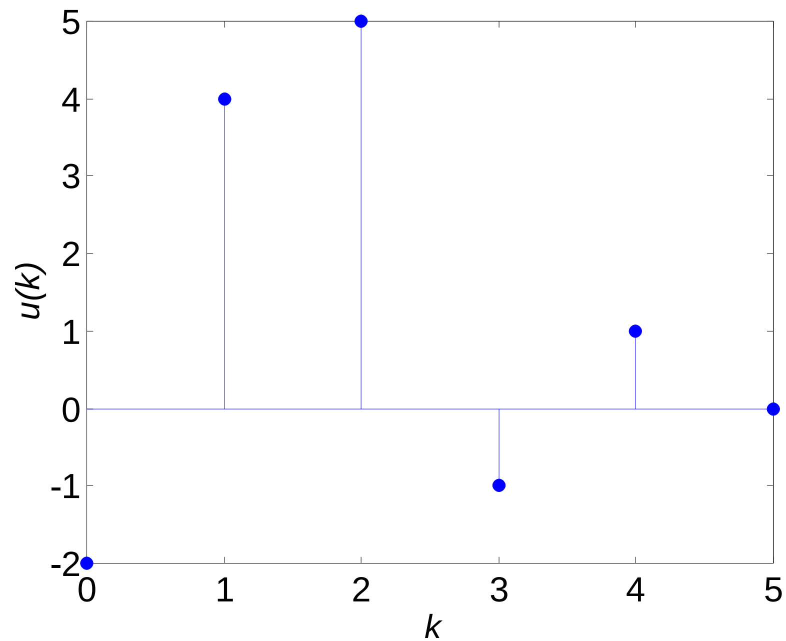
$$y(0) = 1; y(k) = 0, k > 0$$

$$Y(z) = 1z^{0} + 0z^{-1} + 0z^{-2} + \ldots$$

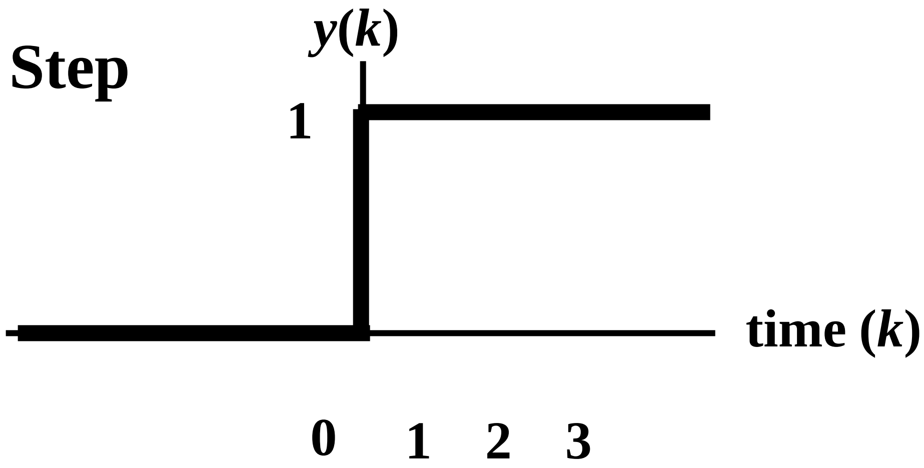
$$= 1$$

$$U(z) = -2 + 4z^{-1} + 5z^{-2} - z^{-3} + z^{-4}$$

This can be viewed as a sum of impulses at time 0, 1, 2, 3, and 4.



Common Signals: Step



$$y(k) = 1, k \geq 0$$

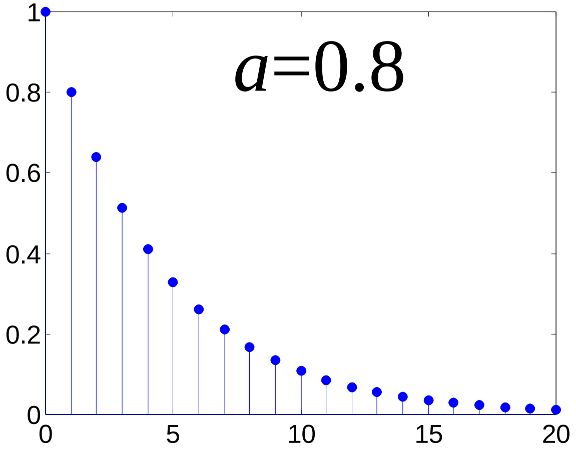
$$Y(z) = 1z^{0} + 1z^{-1} + 1z^{-2} + \ldots$$

$$= \frac{1}{1 - z^{-1}}$$

$$= \frac{z}{z - 1}$$

Common Signals: Geometric

Geometric: $y(k) = a^k$



$$Y(z) = 1 + az^{-1} + a^2 z^{-2} + \ldots$$

$$= \frac{z}{z-a}$$

$$Y(z) = 1 + 0.8z^{-1} + 0.64z^{-2} + \ldots$$

$$= \frac{z}{z-0.8}$$

Properties of *Z*-Transforms of Signals

Signals: $U(z) = u(0)z^0 + u(1)z^{-1} + u(2)z^{-2} + \ldots$

$$V(z) = v(0)z^0 + v(1)z^{-1} + v(2)z^{-2} + \ldots$$

Shift: $zU(z) = u(0)z^1 + u(1)z^0 + u(2)z^{-1} + \ldots$

$$= u(1)z^0 + u(2)z^{-1} + \ldots$$

Delay: $U(z)/z = u(0)z^{-1} + u(1)z^{-2} + u(2)z^{-3} + \ldots$

Scaling: $aU(z) = au(0)z^0 + au(1)z^{-1} + au(2)z^{-2} + \ldots$

$$= z\text{-Transform of } \{au(k)\}$$

Sum of signals: $u(0)z^0 + u(1)z^{-1} + u(2)z^{-2} + \ldots + v(0)z^0 + v(1)z^{-1} + v(2)z^{-2} + \ldots$

$$= (u(0)+v(0))z^0 + (u(1)+v(1))z^{-1} + (u(2)+v(2))z^{-2} + \ldots$$

$$= U(z) + V(z)$$

Poles of a Z-Transform

Definition: Values of *z* for which the denominator is 0

Easy to find the poles of a geometric: $$V(z) = \frac{z}{z-a}$$ Pole is *a*.

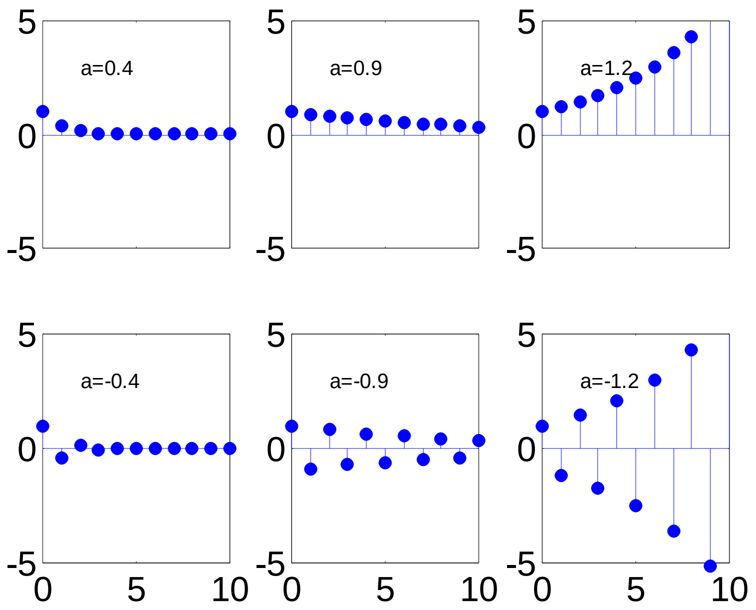
What are the poles of the following Z-Transform? $$Y(z) = \frac{7z^2 - 6z}{z^2 - 1.8z + 0.8}$$

Easy if sum of geometrics $$Y(z) = \frac{5z}{z-1} + \frac{2z}{z-0.8}$$

Poles determine key behaviors of signals

Effect of Pole on the Signal

$$y(k) = a^k \Leftrightarrow \frac{z}{z-a}$$



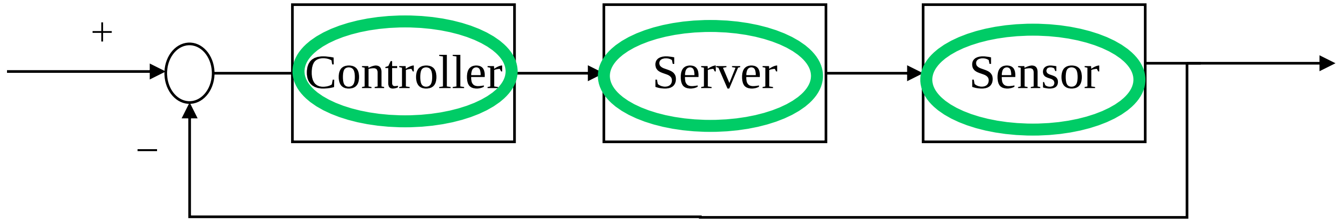
Why?

$$\frac{z}{z-a} = 1 + az^{-1} + a^2 z^{-2} + \ldots \Leftrightarrow (1, a, a^2, \ldots)$$

- What happens when
  - $|a|$ is larger?
  - $|a|>1$?
  - $a<0$?

- $|a|>1$
  - Does not converge
- Larger $|a|$
  - Slower convergence
- $a<0$
  - Oscillates

Part b: Systems & Transfer Functions



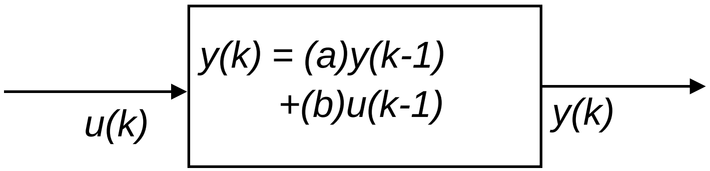
Reference: “Feedback Control of Computer Systems”, Chapter 3.

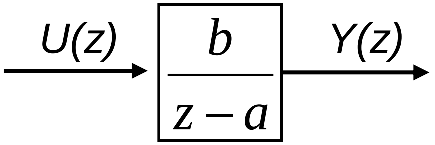
Motivation and Definition

Motivation:

ARX model relates *u(k)* to *y(k)*

(ARX is autoregressive with external input.)





Transfer function expresses this relationship in the z domain

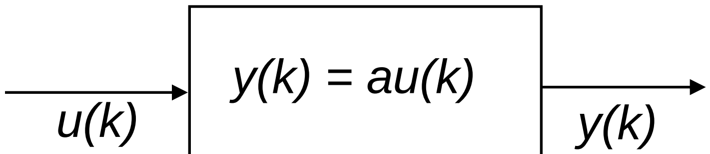
$$G(z) = \frac{\text{Output Signal}}{\text{Input Signal}} = \frac{Y(z)}{U(z)}$$

$$\text{or } Y(z) = G(z)U(z)$$

$$\text{assuming initial conditions are } 0.$$

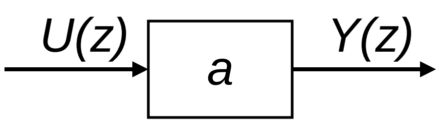
A transfer function is specified in terms of its input and output.

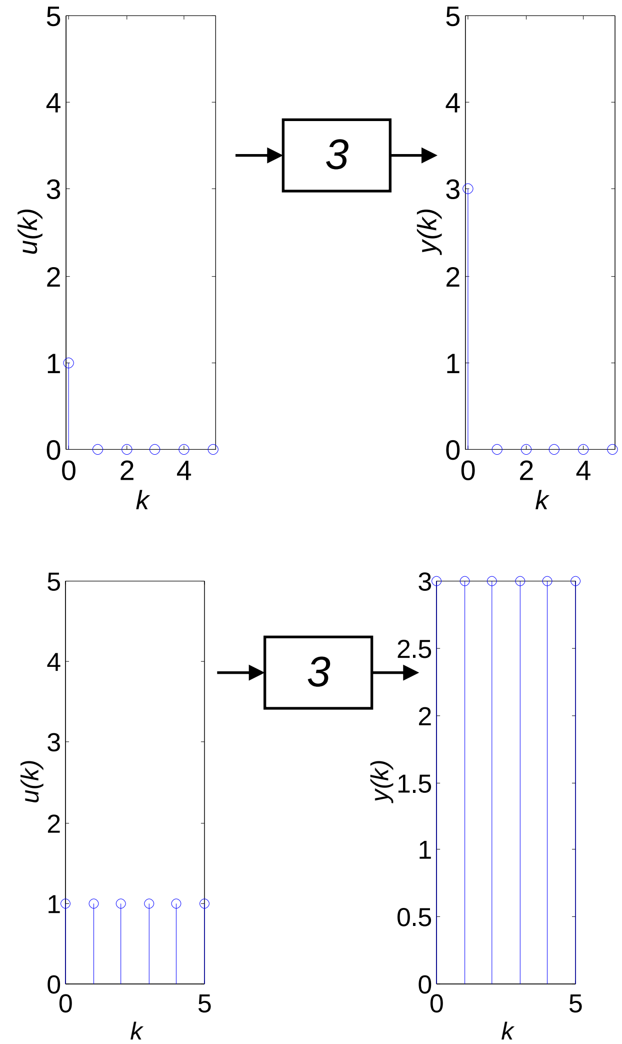
Constant Transfer Function



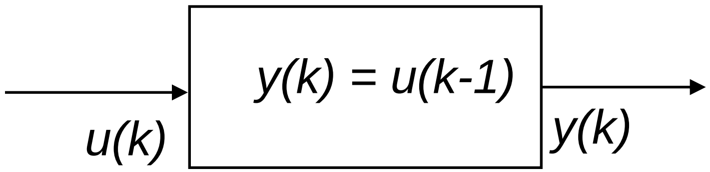
$$Y(z) = aU(z)$$

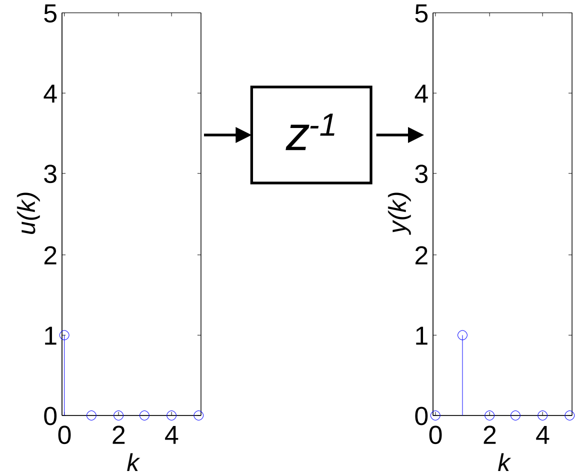
$$G(z) = \frac{Y(z)}{U(z)} = a$$





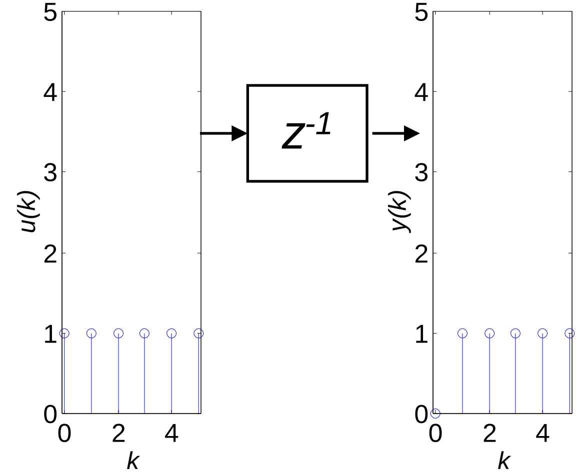
1-Step Time-Delay Transfer Function

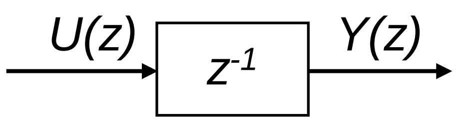




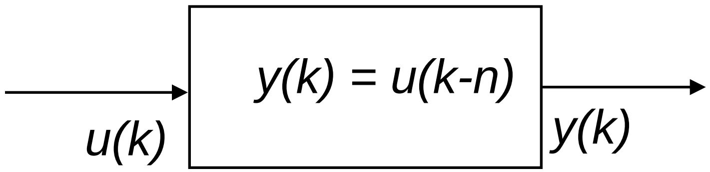
$$Y(z) = z^{-1}U(z)$$

$$G(z) = \frac{Y(z)}{U(z)} = z^{-1}$$



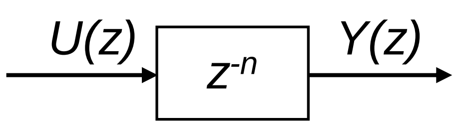


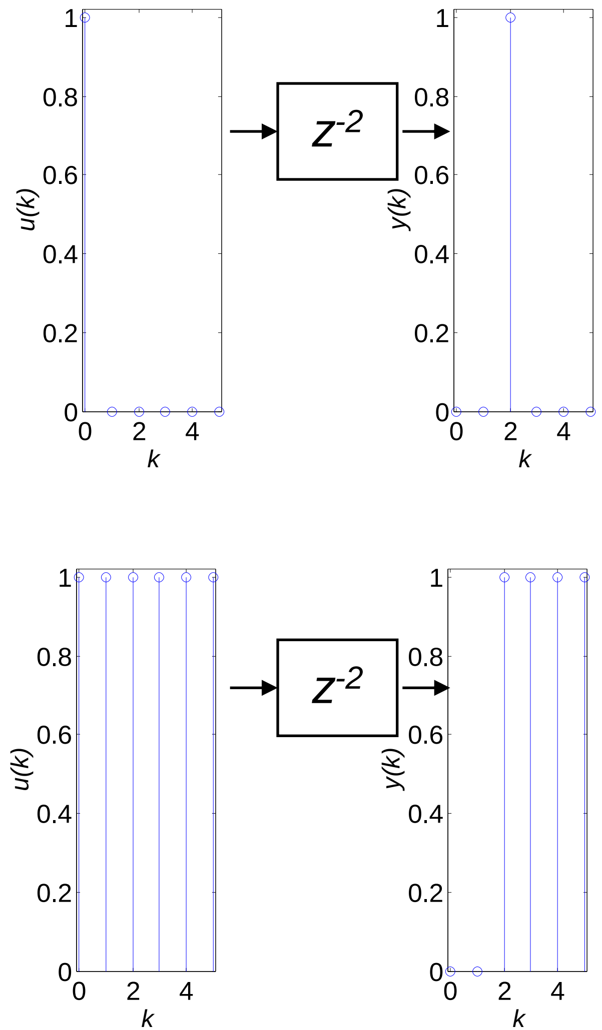
n-Step Time-Delay Transfer Function



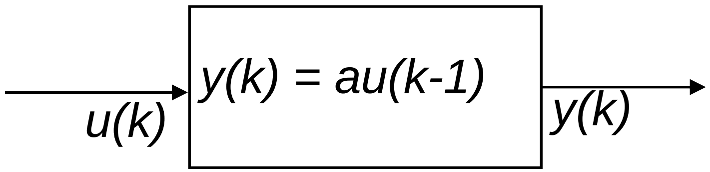
$$Y(z) = z^{-n}U(z)$$

$$G(z) = \frac{Y(z)}{U(z)} = z^{-n}$$



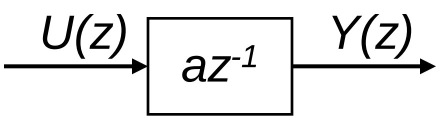


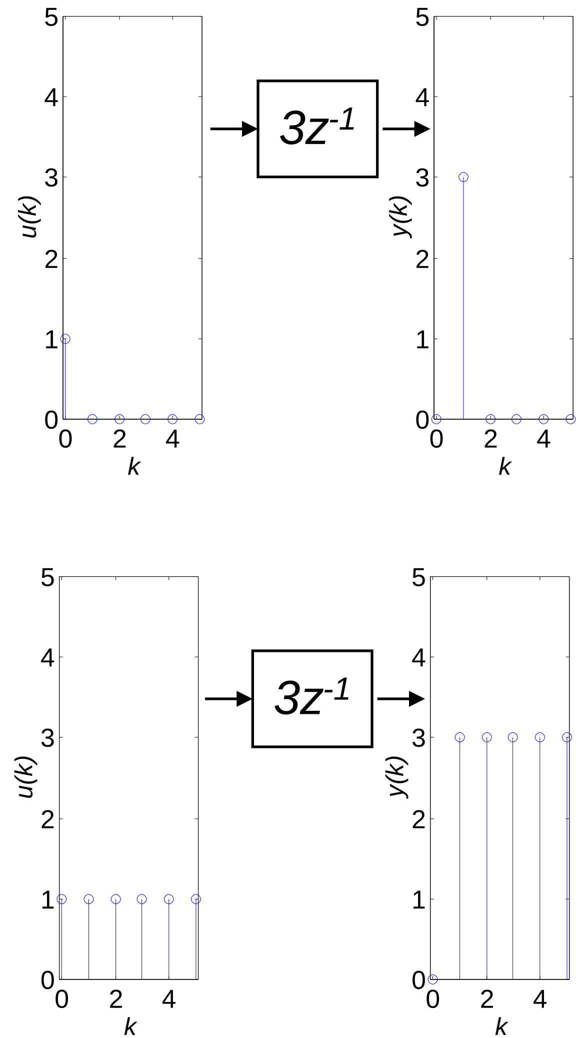
Combining Simple Transfer Functions



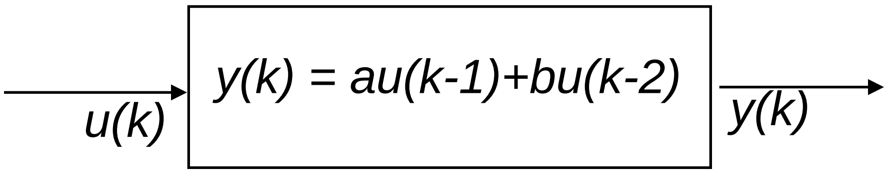
$$Y(z) = az^{-1}U(z)$$

$$G(z) = \frac{Y(z)}{U(z)} = az^{-1}$$



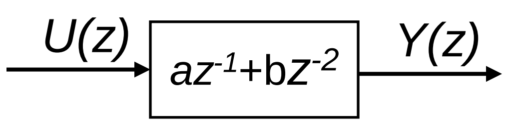


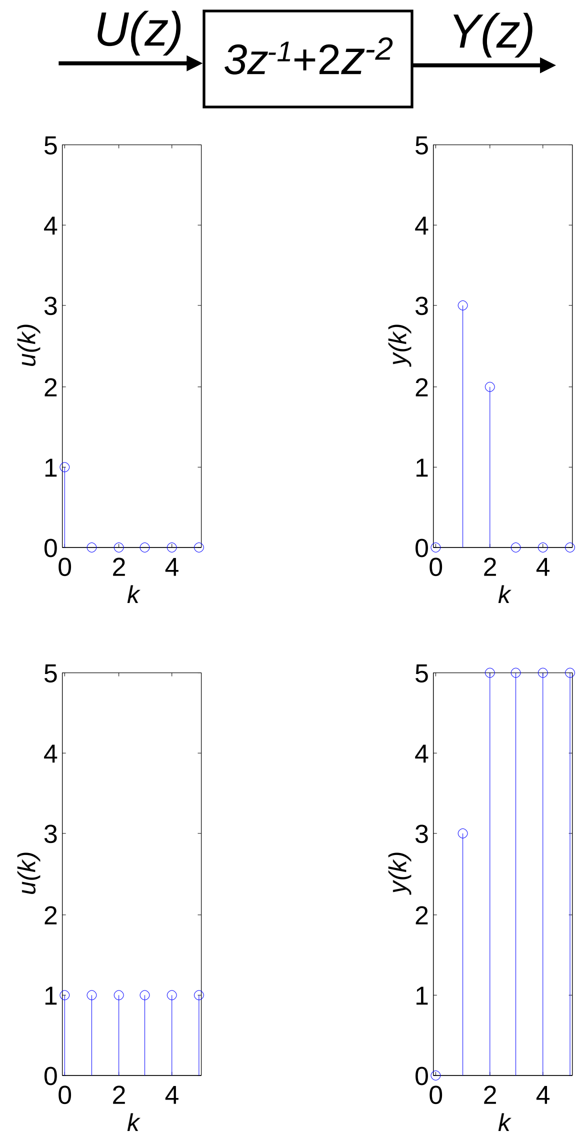
Additional Terms in T.F.



$$Y(z) = (az^{-1} + bz^{-2})U(z)$$

$$G(z) = \frac{Y(z)}{U(z)} = az^{-1} + bz^{-2}$$

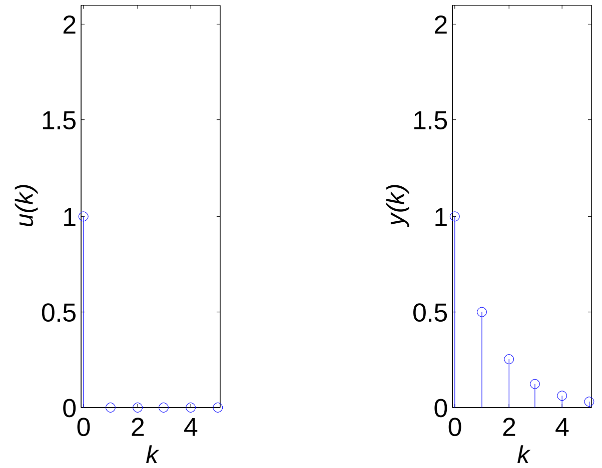




Geometric Sum of T.F.

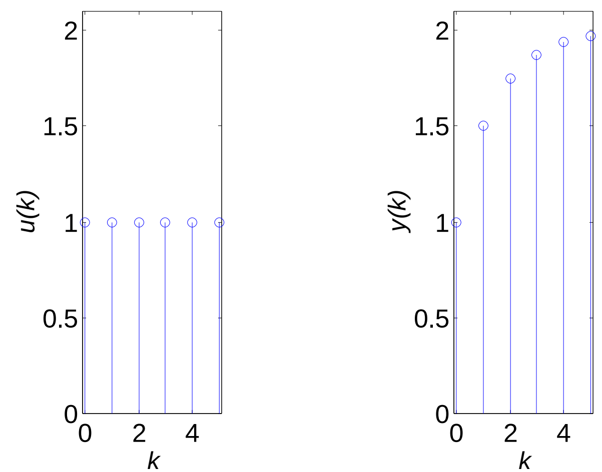
U(z) → z/(z-0.5) → Y(z)

u(k) → $y(k) = u(k) + au(k-1) + a^2u(k-2) + \dots$
$= ay(k-1) + u(k)$ → y(k)



$$G(z) = \frac{Y(z)}{U(z)} = (1 + az^{-1} + a^2 z^{-2} + \dots)$$

U(z) → $1 + az^{-1} + a^2z^{-2} + \dots$ → Y(z)

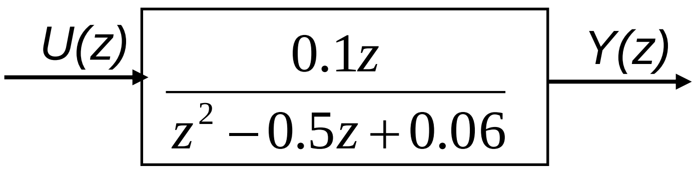


$$G(z) = \frac{Y(z)}{U(z)} = 1 + az^{-1} + a^2 z^{-2} + \dots = \frac{z}{z-a}$$

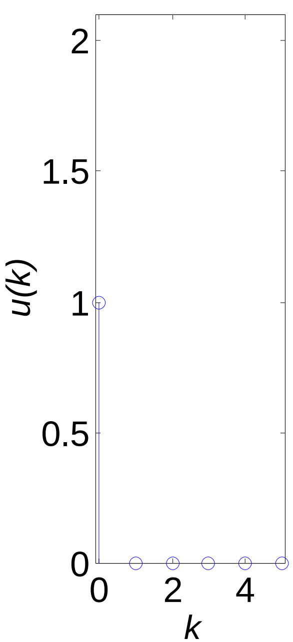
U(z) → z/(z-a) → Y(z)

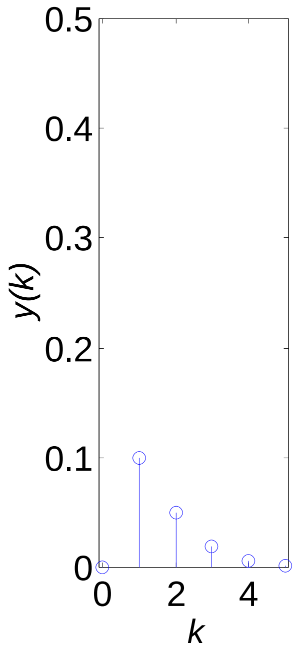
Complicated Transfer Functions

<u>Decompose into a sum of geometrics</u>



$$G(z)=\frac{0.1z}{z^2-0.5z+0.06}$$
$$=\frac{z}{z-0.3}-\frac{z}{z-0.2}$$
$$=1+0.3z^{-1}+0.09z^{-2}+\ldots-1-0.2z^{-1}-0.04z^{-2}+\ldots$$
$$=0.1z^{-1}+0.05z^{-2}+\ldots$$



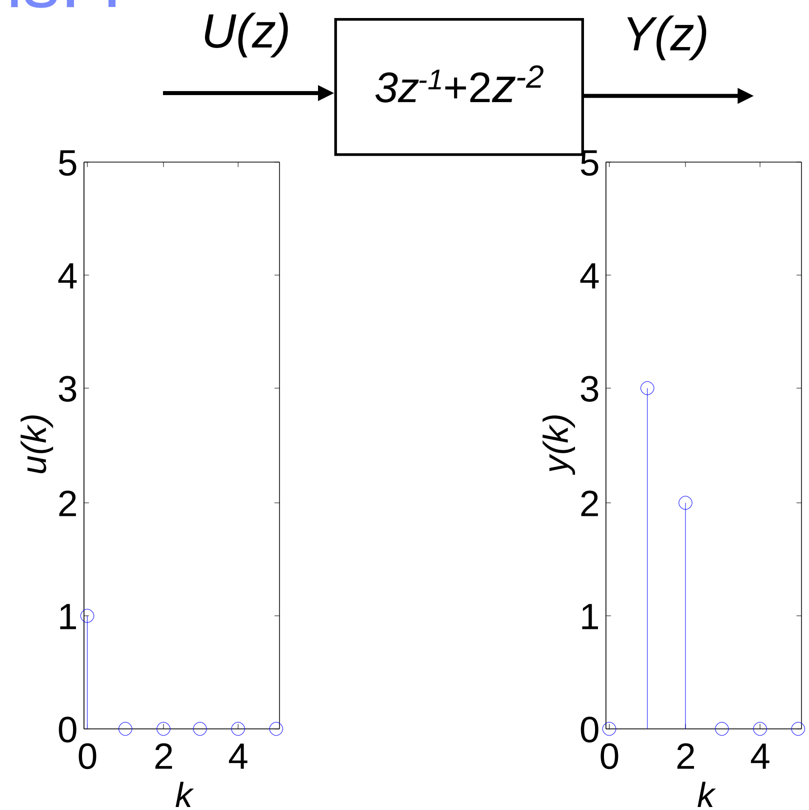


- **<u>Partial fraction expansion</u>** allows rational polynomials to be decomposed into a sum of geometrics
- Poles of the original polynomial are the poles of the geometrics

Interpreting Transfer Functions: I

<u>Signal generated by an impulse input</u>

Example:



$$G(z) = \frac{Y(z)}{U(z)}$$

$$Y(z) = G(z)U(z) = G(z)(1) = G(z)$$

Interpreting Transfer Functions: II

Specifies an ARX model

Given a transfer function:

$$G(z)=\frac{Y(z)}{U(z)}=\frac{z}{z-0.5}$$

$$\text{So, } Y(z)(z-0.5)=zU(z) \text{ or } zY(z)=0.5Y(z)+zU(z)$$

Recall that:

$$aY(z)\Leftrightarrow ay(k)$$

$$zY(z)\Leftrightarrow y(k+1)$$

Which gives us:

$$y(k+1)=0.5y(k)+u(k+1) \text{ which is equivalent to}$$

$$y(k)=0.5y(k-1)+u(k)$$

This means that transfer functions are trivial to simulate!

Constructing Transfer Functions

- Given a ARX model, how do we construct its transfer function?
- Method: Term by term conversion from time domain to *z* Domain: (1) substitute for *z* expressions and (2) factor to obtain the ratio of output to *z*-Transforms.
- Example

$$\text{Given } y(k) = (0.43)y(k-1) + (0.47)u(k)$$

$$y(k) \Leftrightarrow Y(z)$$

$$y(k-1) \Leftrightarrow z^{-1}Y(z)$$

$$u(k) \Leftrightarrow U(z)$$

$$\text{Substitute }: Y(\mathrm{z}) = (0.43)z^{-1}Y(z) + (0.47)U(z)$$

$$\text{Factor}: \frac{Y(z)}{U(z)} = \frac{(0.47)z}{z-0.43}$$

A Pop Quiz

What is the transfer function for

$$y(k+1) = (0.8)y(k) + (0.72)w(k) - (0.66)w(k-1)$$

Hint : $y(k+1) \Leftrightarrow zY(z)$

Step 1 : Substitute

$$zY(z) = (0.8)Y(z) + (0.72)W(z) - (0.66)z^{-1}W(z)$$

Step 2 : Factor

$$\frac{Y(z)}{W(z)} = \frac{(0.72)z - 0.66}{z^2 - (0.8)z}$$

Test Your Knowledge Again

Given the transfer function $G(\mathrm{z}) = \dfrac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_0}{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0}$

Why must it be that $n \geq m$?

Write the ARX model :

$$a_n y(k+n) + a_{n-1} y(k+n-1) + \ldots + a_0 = b_m u(k+m) + b_{m-1} u(k+m-1) + \ldots + b_0$$

Adjust tim e so that $k+n \rightarrow k$

$$a_n y(k) + a_{n-1} y(k-1) + \ldots + a_0 = b_m u(k+m-n) + b_{m-1} u(k+m-n-1) + \ldots + b_0$$

If $m > n$, then $y(k)$ is a function of one or more $u(k+m-n)$ in the future!

Psychic System!

Poles of a Transfer Function

<u>Poles: Values of *z* for which the denominator is 0.</u>

Example:

$$H(z) = \frac{0.1z}{z^2 - 0.5z + 0.06} = \frac{z}{z-0.3} - \frac{z}{z-0.2}$$

Poles: 0.3, 0.2

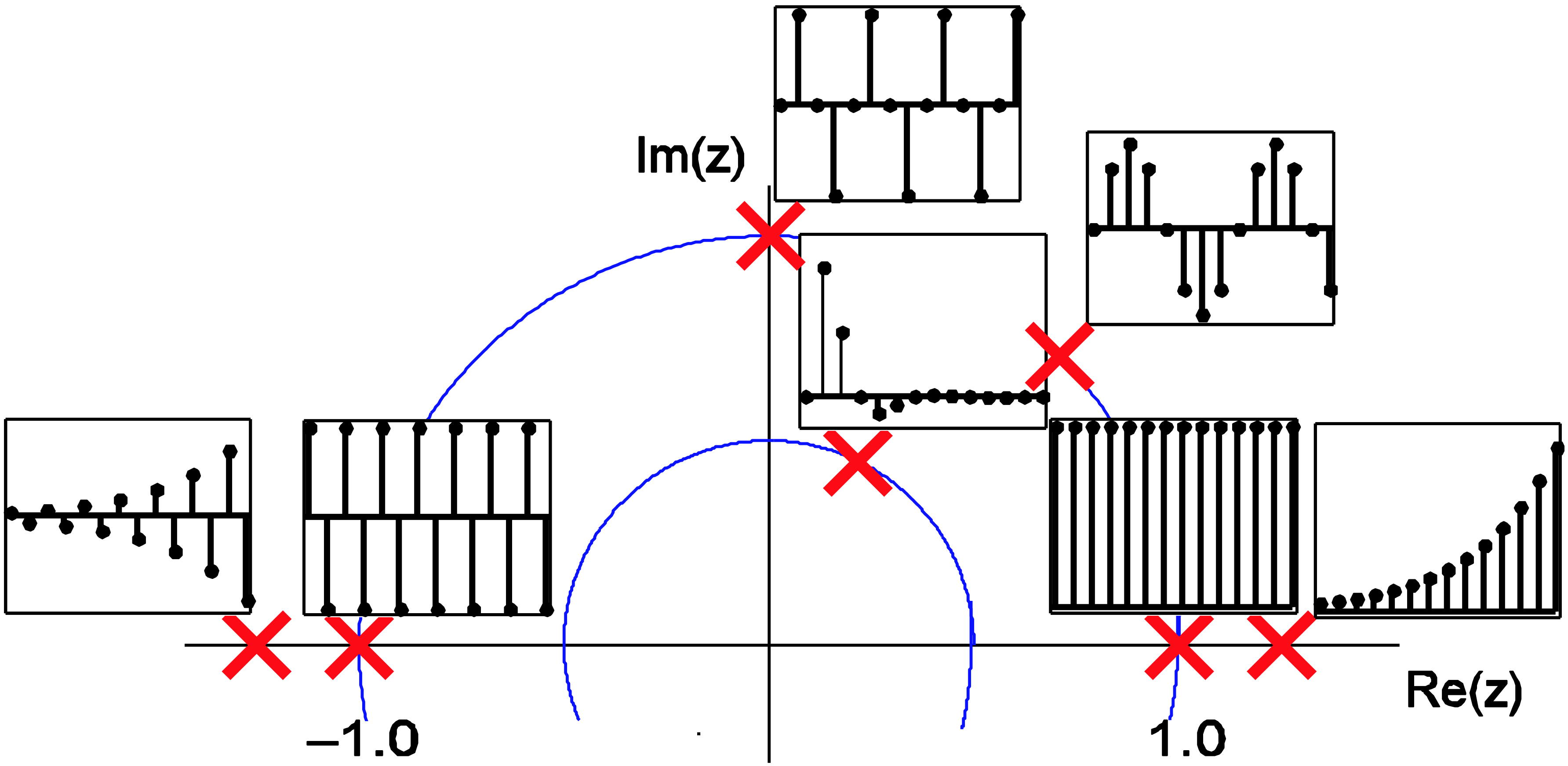
Poles

- Determine stability
- Major effect on settling time, overshoot
- **<u>Dominant pole</u>** – pole that determines the transient response

$$G(z) = \frac{z}{z-a} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \ldots$$

- $|a|>1$
  - Does not converge
- $|a|<1$ but large
  - Slower convergence
- $a<0$
  - Oscillates

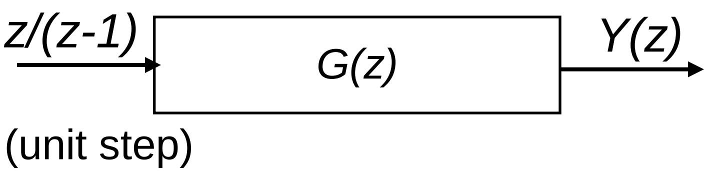
Almost All You'll Ever Need to Know About Poles



$$G(z) = \frac{z}{z-a} = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \ldots$$

Settling Time (k_S) of a System

Definition and result: Time until an input signal is within 2% of its steady state value



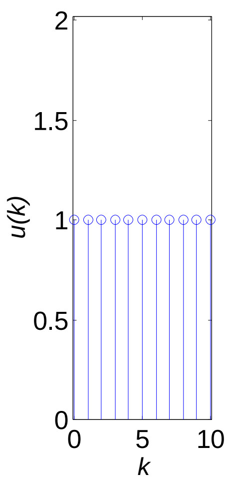
$$k_S \approx \frac{-4}{\ln|a|}, \text{ where } |a| \text{ is the largest pole of } G(z),$$

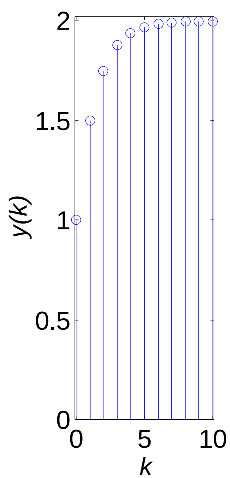
$$\text{with equality for } G(z) = \frac{z}{z-a} = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \ldots$$

Examples:

$$G(z) = \frac{z}{z-0.5}$$

$$k_S \approx \frac{-4}{\ln 0.5} \approx 6$$

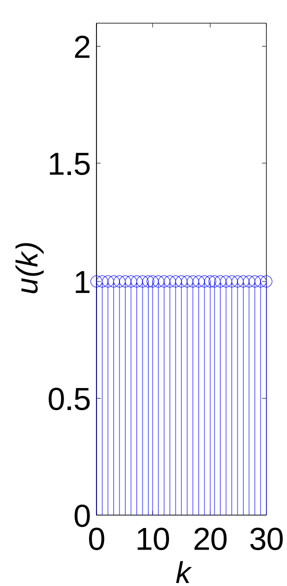


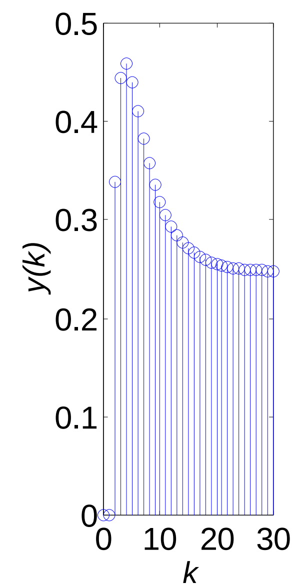


$$G(z) = \frac{0.34z - 0.31}{z^3 - 1.23z^2 + 0.34z},$$

$$\text{poles}: 0, 0.43, 0.8$$

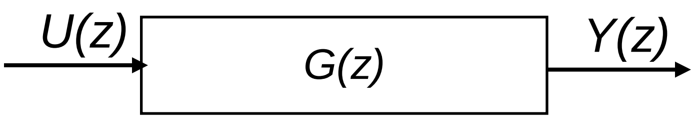
$$k_S \approx \frac{-4}{\ln 0.8} \approx 18$$





Steady State Gain (ssg) of a Transfer Function

Steady state gain is the steady state output in response to a step input.

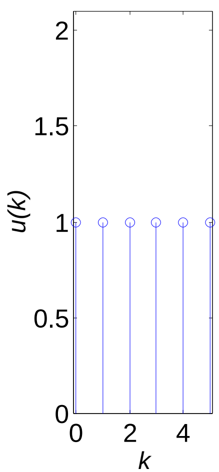


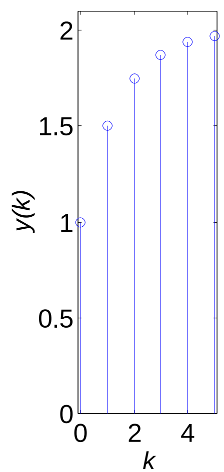
ssg of *G(z)* is $\dfrac{y(\infty)}{u(\infty)} = G(1)$

where *U(z)* is a step input.

Example:

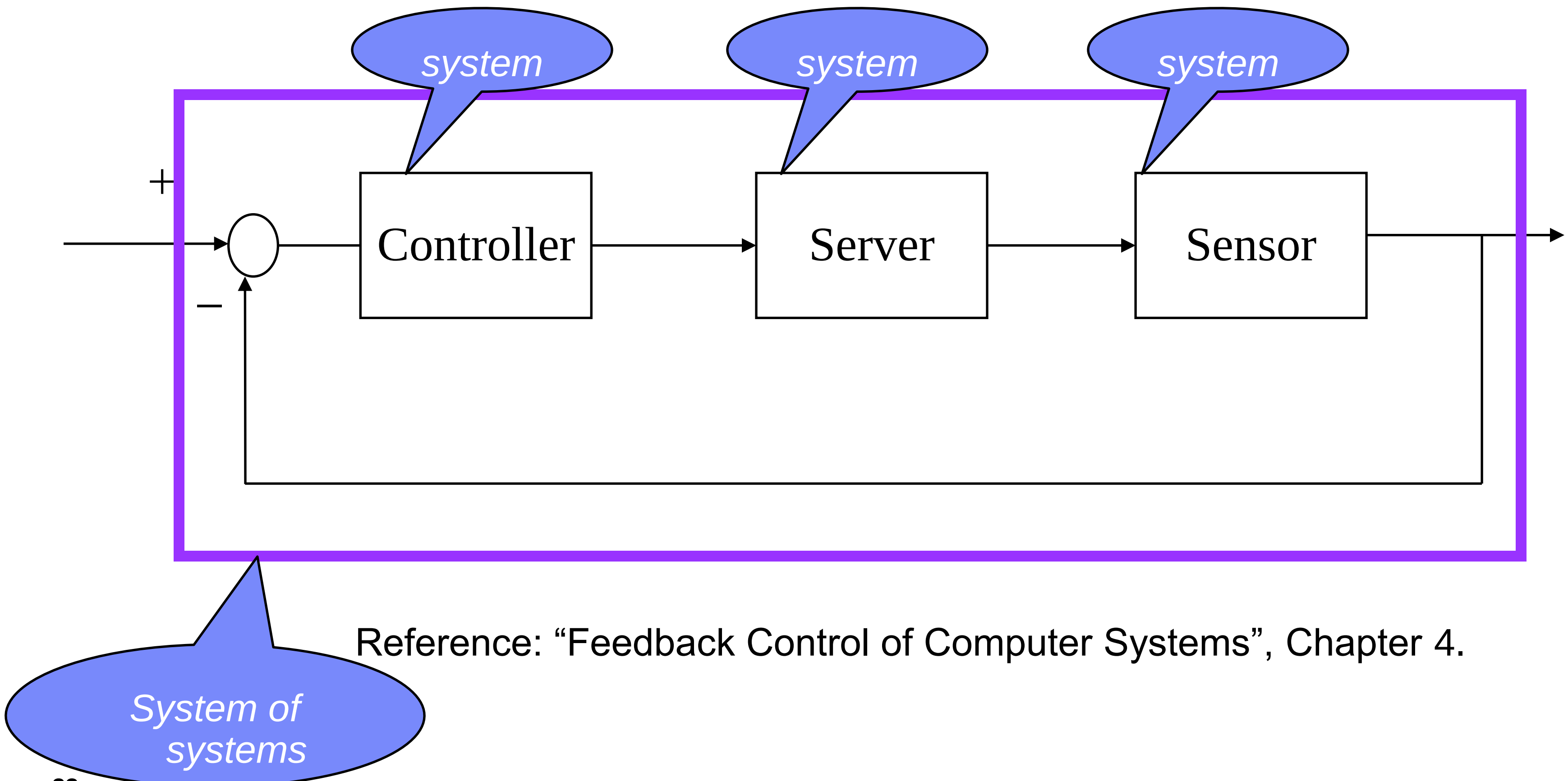
$$u(\infty) = 1 \qquad G(z) = \frac{z}{z-0.5} \qquad y(\infty) = 2$$





$$\frac{y(\infty)}{u(\infty)} = \frac{2}{1} = 2 = G(1)$$

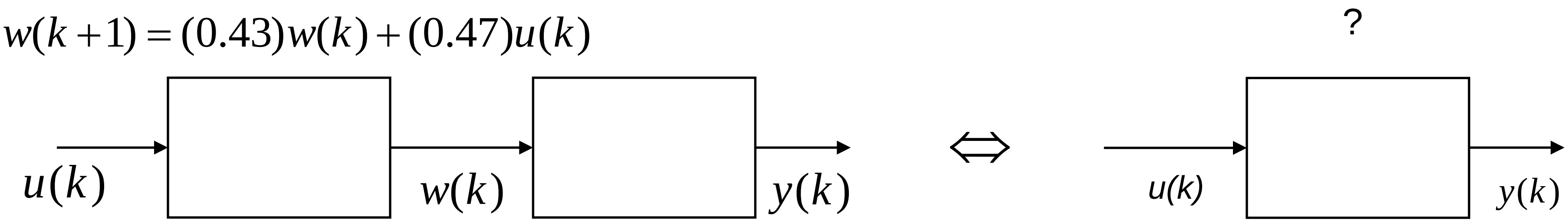
M2c – Theory
Composition of Systems



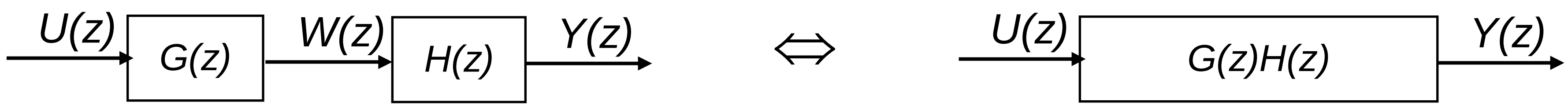
Reference: “Feedback Control of Computer Systems”, Chapter 4.

Transfer Functions In Series

$$w(k+1) = (0.43)w(k) + (0.47)u(k)$$



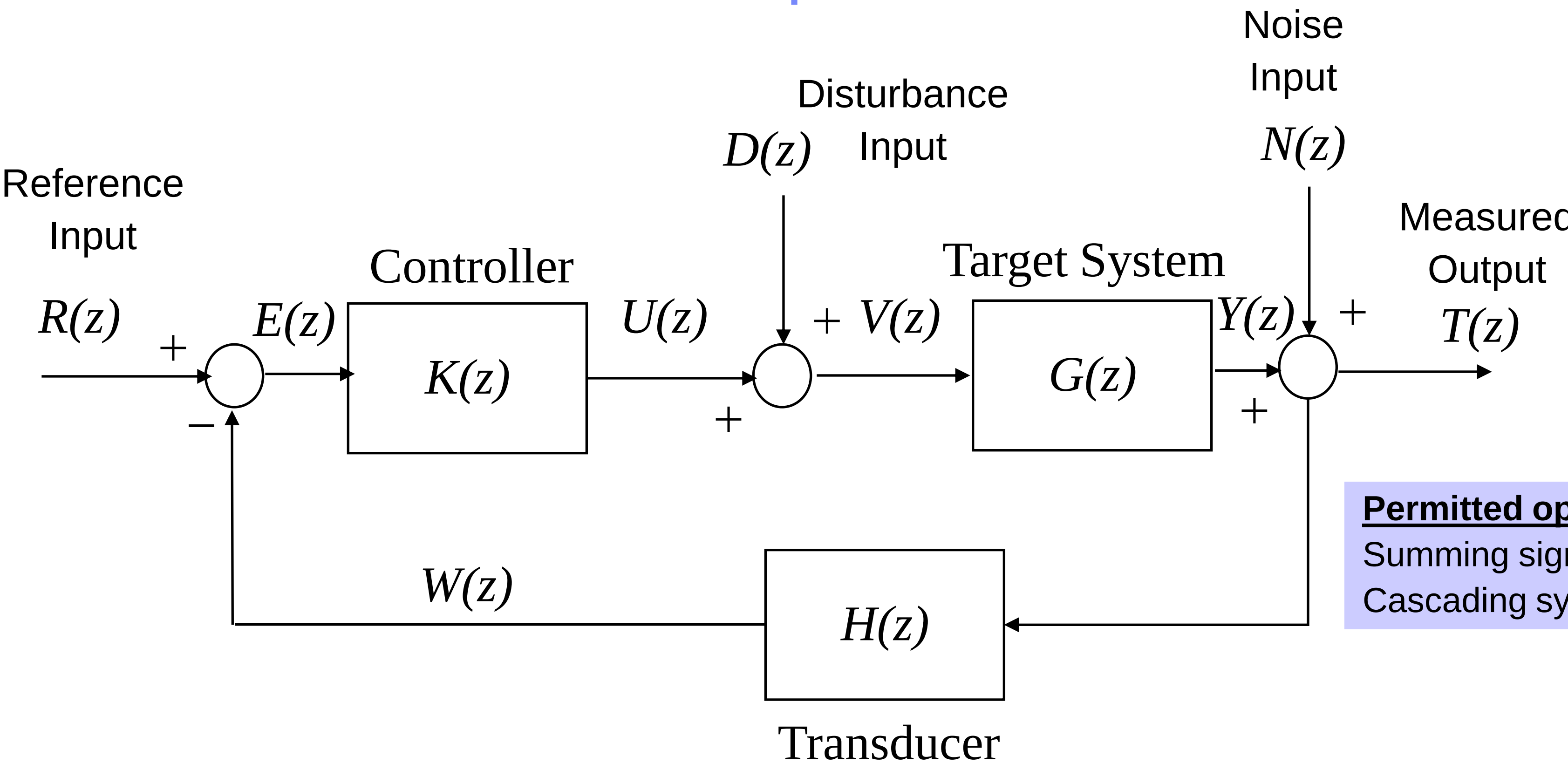
$$y(k+1) = 0.8y(k) + 0.72w(k) - 0.66w(k-1)$$



$$\frac{Y(z)}{U(z)} = \frac{W(z)}{U(z)}\frac{Y(z)}{W(z)} = G(z)H(z)$$

T.F. provide an easy way to analyze the behavior of complex structures.

Canonical Feedback Loop



Permitted operations

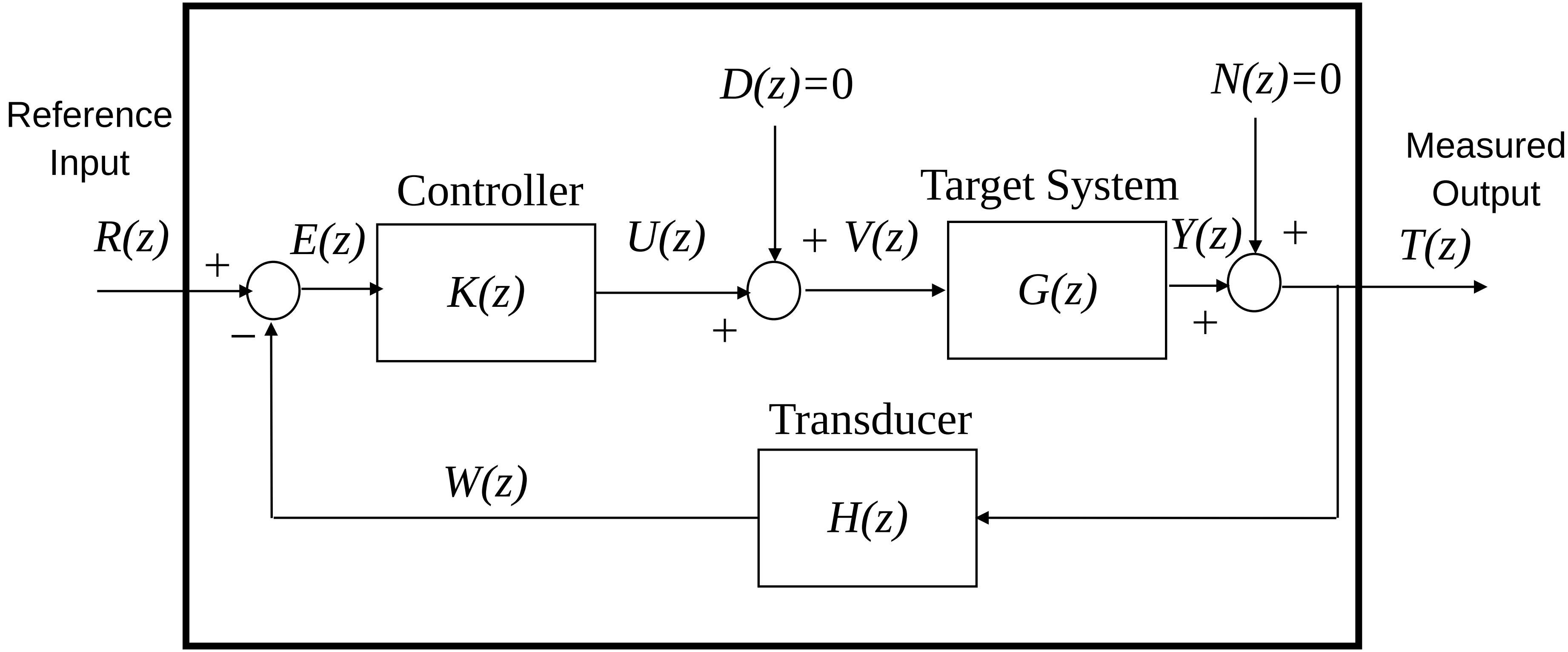
Summing signals

Cascading systems

Want to analyze characteristics of the entire system: its stability, settling time, and accuracy (ability to achieve the reference input).

It's all done with transfer functions!

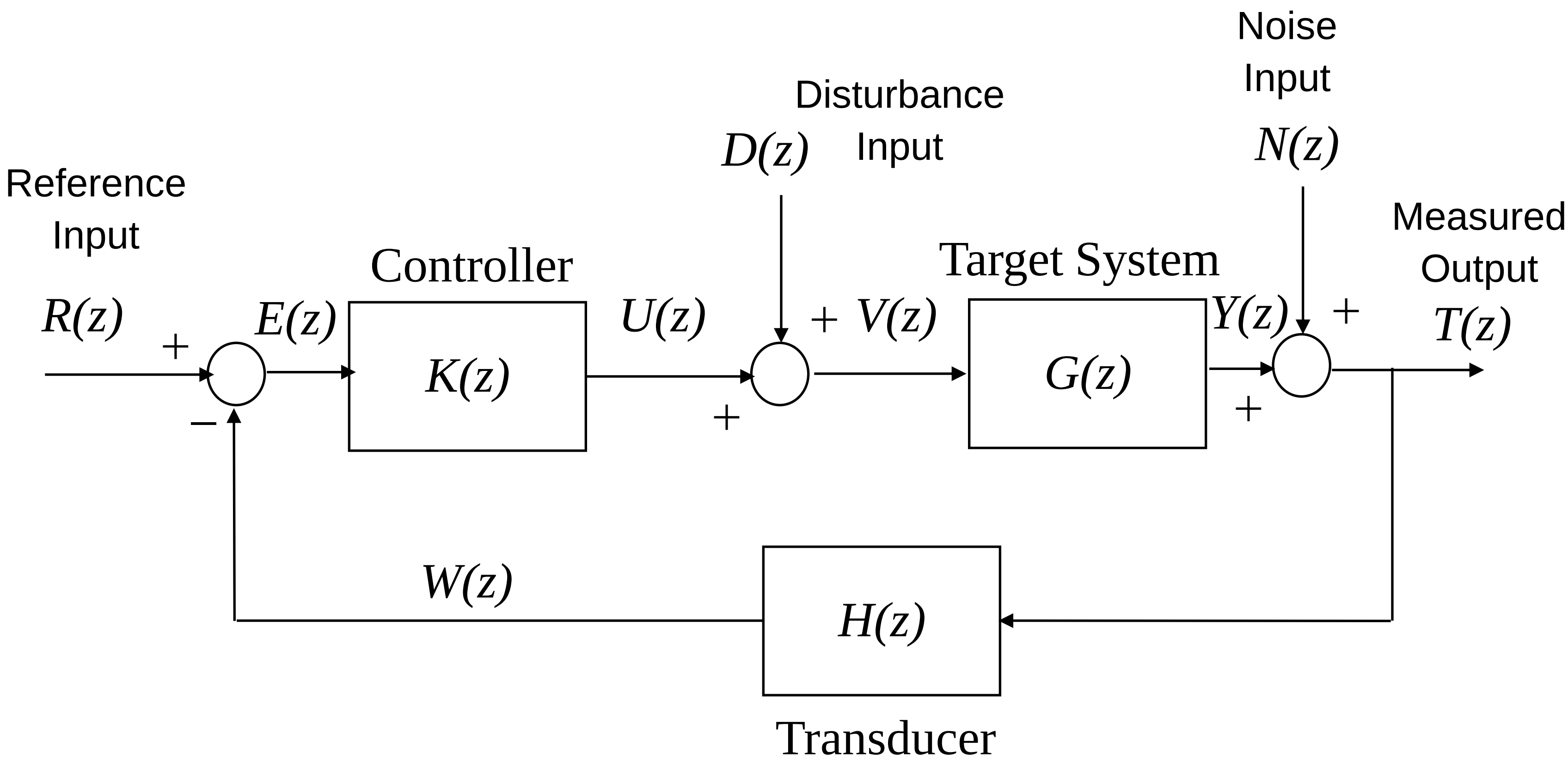
$$F_R(z) = \frac{T(z)}{R(z)}$$



View the dark rectangle as a large transfer function $F_R(z)$ with input $R(z)$ and output $T(z)$.

- System is stable if the largest pole of $F_R(z)$ has an absolute value that is less than 1
- System is accurate if $t(n)=r(n)$ for large n, or $F_R(1)=1$
- System settling time is short if the poles of $F_R(z)$ have a small absolute value
- System has oscillations if there are poles of $F_R(z)$ that are negative or imaginary

Canonical Feedback Loop Has Many T.F.



$F_R(z)$

Transfer function from the reference input to the measured output

$F_D(z)$

Transfer function from the disturbance input to the measured output

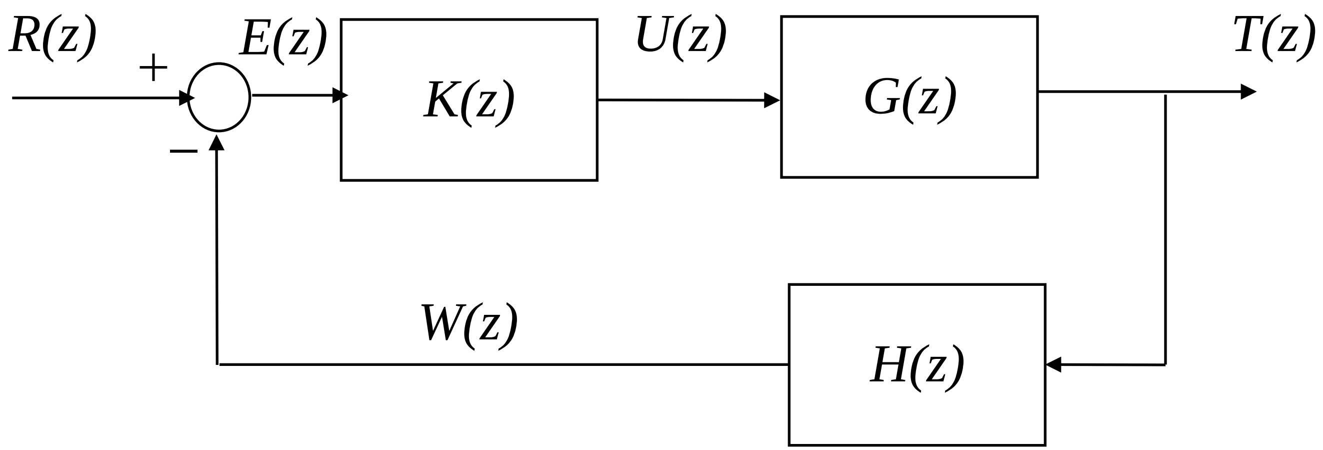
$F_N(z)$

Transfer function from the noise input to the measured output

Computing $F_R(z)$

The only non-zero input is *R(z)*.

Simplified block diagram
since *D(z)=0=N(z)*

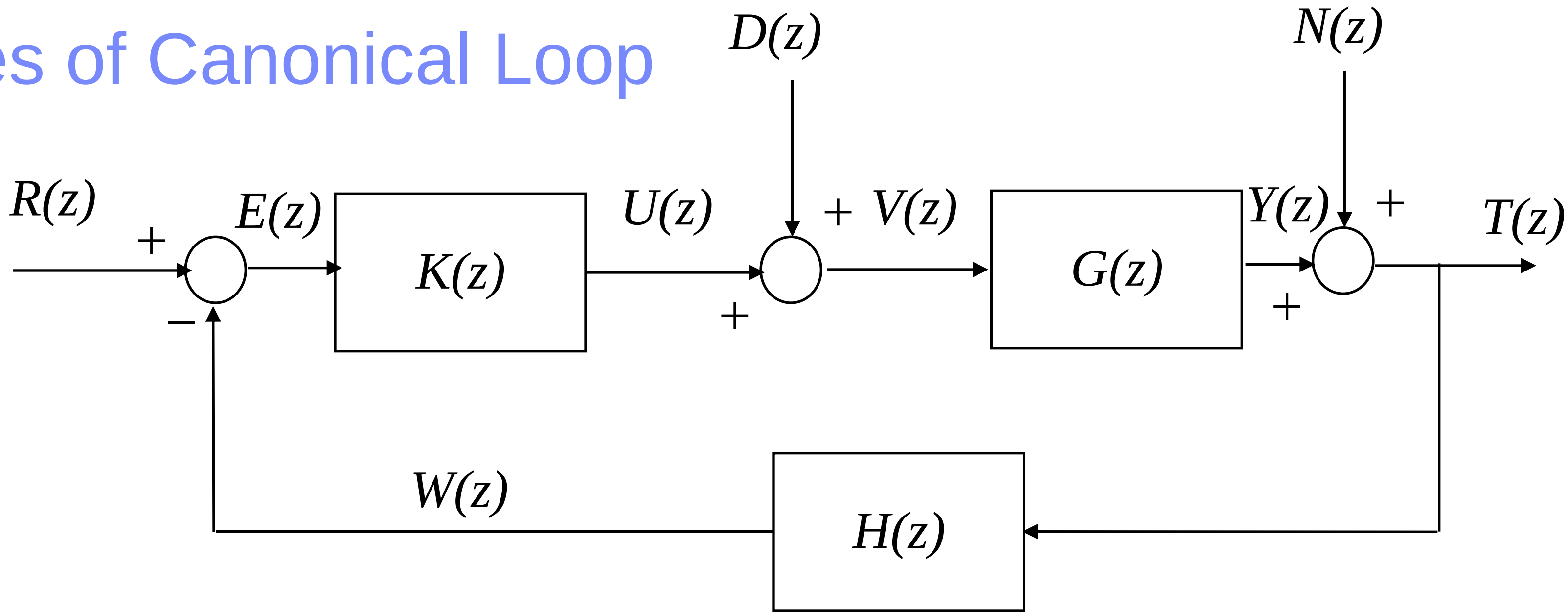


A set of equations relates *R(z)* to *T(z)* based on our previous results

W(z) = *H(z)T(z)* by the definition of a transfer function.
E(z) = *R(z)*-*W(z)* since this is an addition of signals.
T(z) = *E(z)K(z)G(z)* since *K(z)* and *G(z)* are in series.
T(z) = *(R(z)-H(z)T(z))K(z)G(z)* by substitution.

$$F_R(z) = \frac{T(z)}{R(z)} = \frac{K(z)G(z)}{1+K(z)G(z)H(z)}$$

Properties of Canonical Loop



Reference to Output

$$F_R(z) = \frac{T(z)}{R(z)} = \frac{K(z)G(z)}{1+K(z)G(z)H(z)}$$

Disturbance to Output

$$F_D(z) = \frac{T(z)}{D(z)} = \frac{G(z)}{1+K(z)G(z)H(z)}$$

Noise to Output

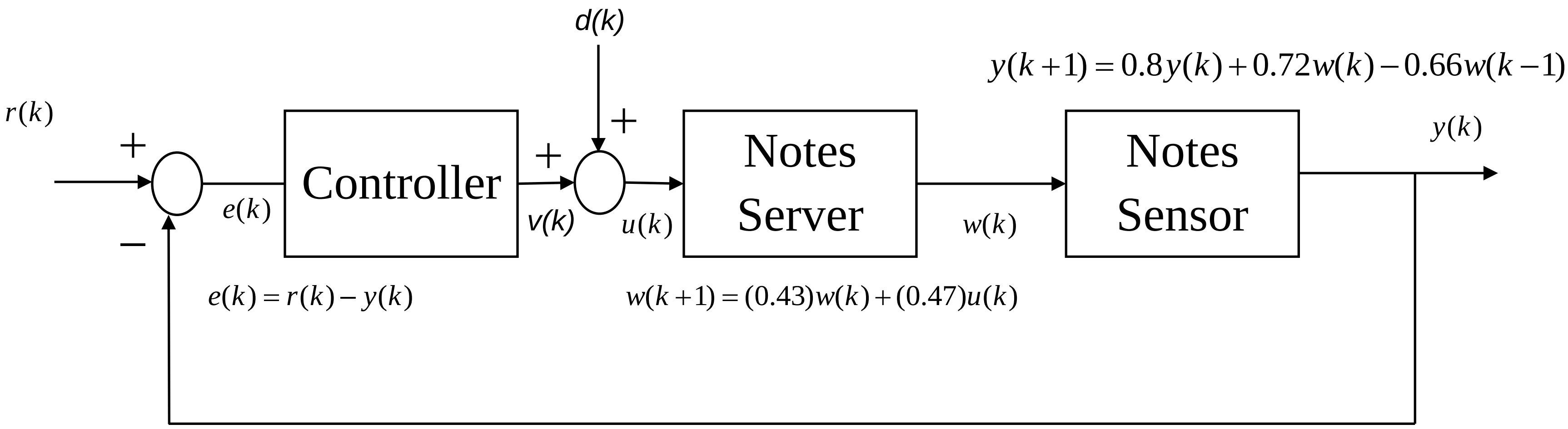
$$F_N(z) = \frac{T(z)}{N(z)} = \frac{1}{1+K(z)G(z)H(z)}$$

What can we say about the stability and settling times of these three transfer functions?
They are the same!

When is the system accurate in the sense that *T(z)*=*R(z)*? $\boldsymbol{F_R(1)=1}$

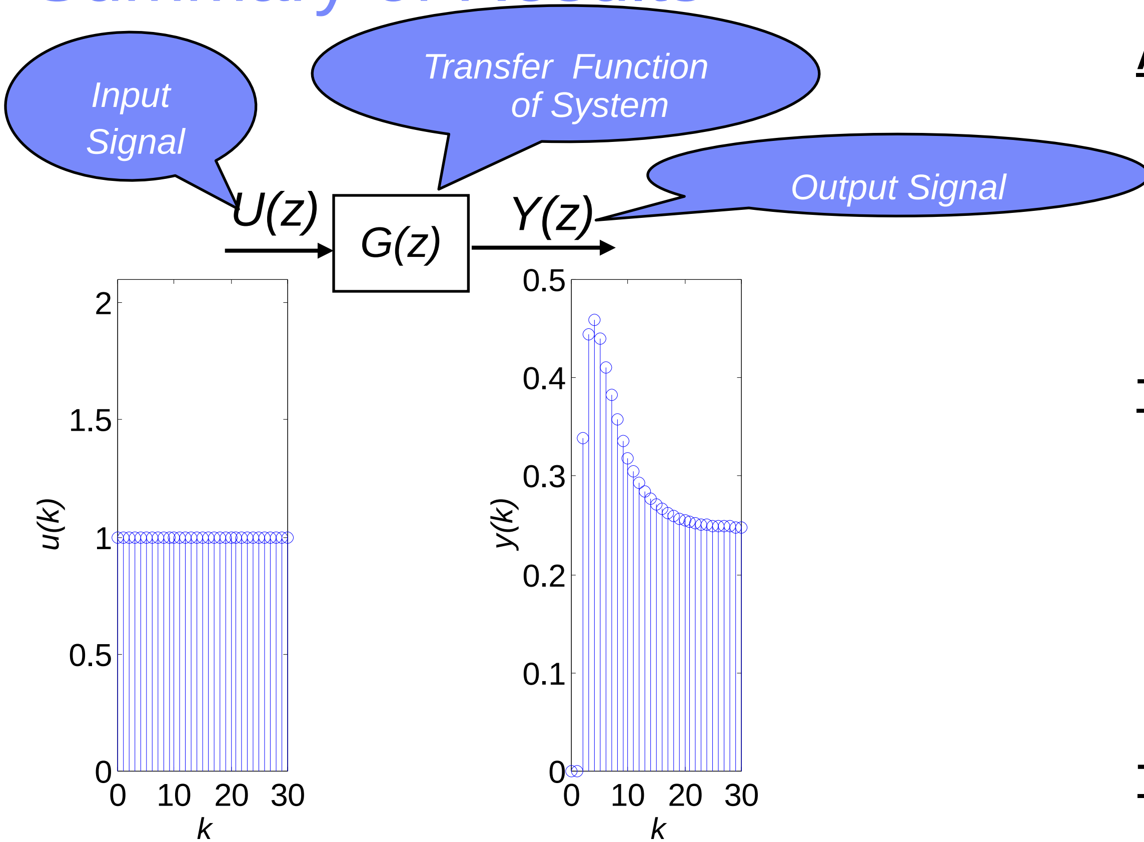
When is the system robust to disturbances and noise? $\boldsymbol{F_D(1)=0=F_N(1)}$

Lab 3: Effect of a Disturbance (Try this later on)

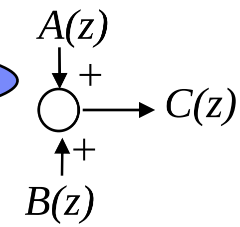


- Model is in file CTShortClass, tab 3 (Notes + Sensor + Disturbance)

Summary of Results

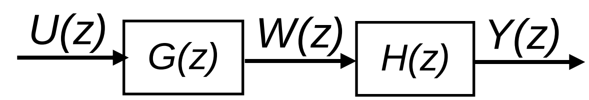


Adding signals:

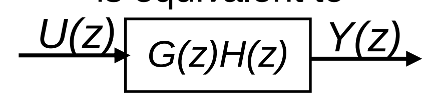


{c(k)=a(k)+b(k)} has Z-Transform *A(z)+B(z).*

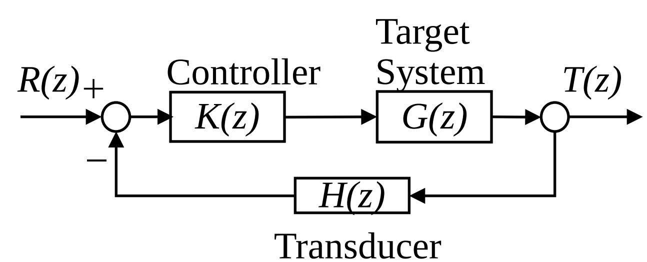
Transfer functions in series



is equivalent to



Transfer function of a feedback loop



Stable system if |*a*|<*1*, where *a* is the largest pole of *G(z)*

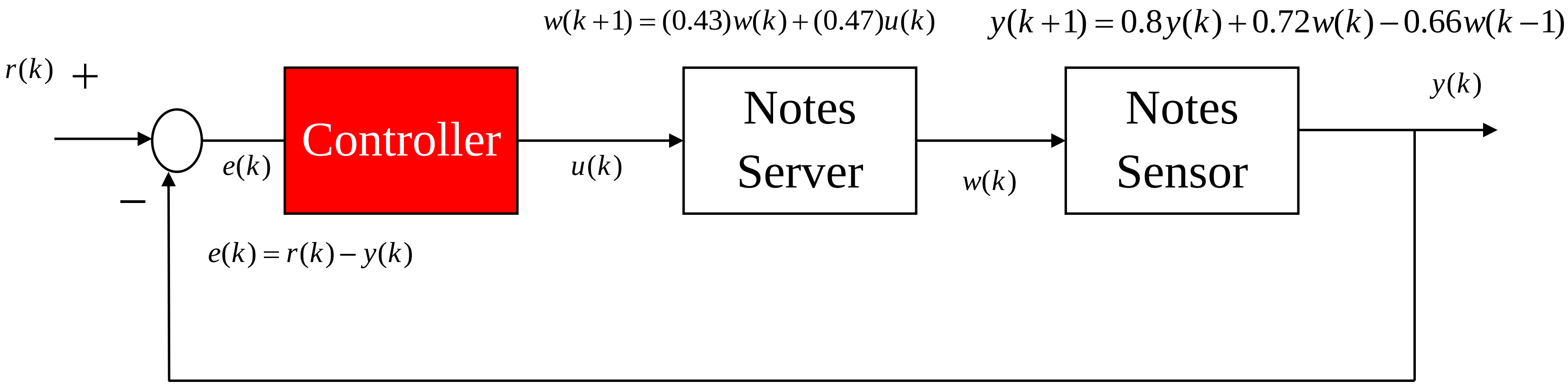
$$\text{Settling time} \approx \frac{-4}{\ln|a|}, \text{ where } |a| \text{ is the largest pole of } G(z)$$

Steady state gain of *G(z)* is $\dfrac{y(\infty)}{u(\infty)} = G(1)$

$$F_R(z) = \frac{T(z)}{R(z)} = \frac{K(z)G(z)}{1 + H(z)K(z)G(z)}$$

Lab:
Control Analysis

Motivating Example

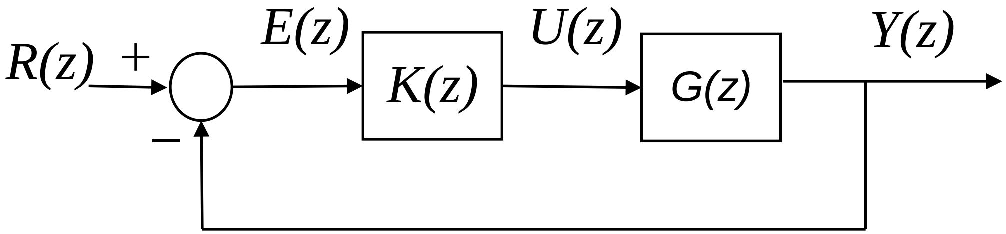


<u>The problem</u>

Design a control system that is stable, accurate, settles quickly, and has small overshoot.

Return to the spreadsheet.

Basic Controllers



Proportional (P) Control

$$u(k) = K_P e(k)$$

$$zU(z) = K_P E(z)$$

$$K(z) = \frac{U(z)}{E(z)} = K_P$$

Integral (I) Control

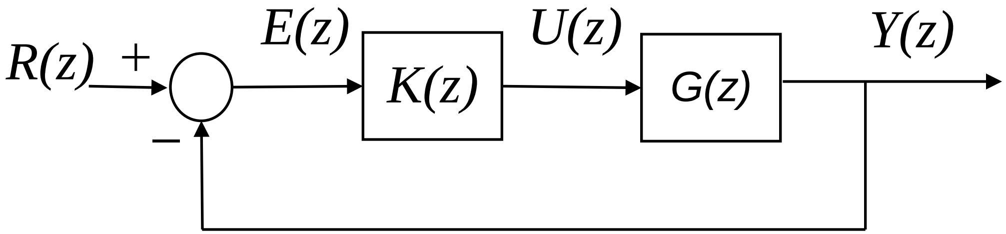
$$u(k+1) = u(k) + K_I e(k+1)$$

$$zU(z) = U(z) + K_I zE(z)$$

$$K(z) = K_I \frac{z}{z-1}$$

K_P and K_I are called ***control gains***.

Analysis for P Control



$$F_R(z) = \frac{Y(z)}{R(z)} = \frac{K(z)G(z)}{1+K(z)G(z)}$$

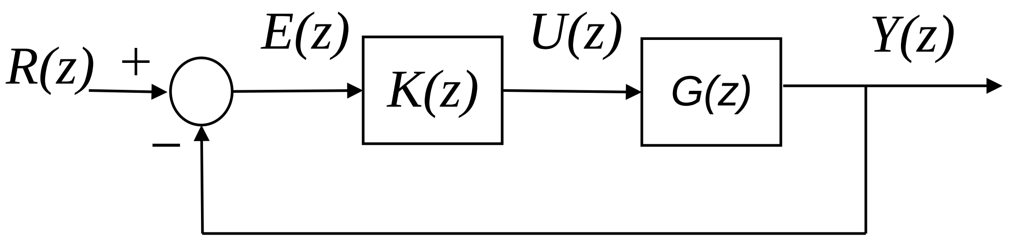
$$F_R^P(z) = \frac{Y(z)}{R(z)} = \frac{K_P \dfrac{0.47}{z-0.43}}{1+K_P \dfrac{0.47}{z-0.43}}$$

$$= \frac{0.47K_P}{(z-0.43)+0.47K_P}$$

$$0 = z - 0.43 + 0.47K_P$$

$$p_P = 0.43 - 0.47K_P$$

Analysis for I Control



$$F_R(z) = \frac{Y(z)}{R(z)} = \frac{K(z)G(z)}{1+K(z)G(z)}$$

$$F_R^I(z) = \frac{Y(z)}{R(z)} = \frac{K_I \frac{z}{z-1}\frac{0.47}{z-0.43}}{1+K_I \frac{z}{z-1}\frac{0.47}{z-0.43}}$$

$$= \frac{0.47K_I z}{(z-1)(z-0.43)+0.47K_I z}$$

$$= \frac{0.47K_I z}{z^2+(0.47K_I-1.43)z+0.43}$$

$$p_I = \frac{1.43-0.47K_I \pm \sqrt{(0.47K_I-1.43)^2-1.72}}{2}$$

Characteristic Equation

$$(0.47K_I-1.43)^2-1.72=0$$

BACKUP

Summary of Lab 2: P vs. I Control

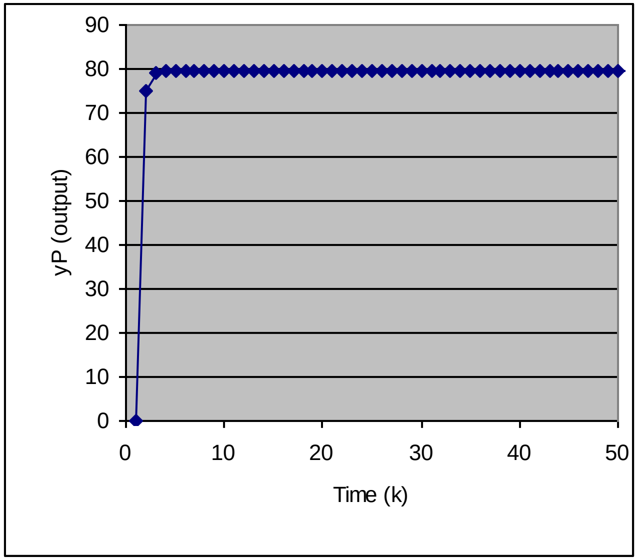
Proportional (P) Control $K(z) = K_P$

eP(k) = r(k)-yP(k)

uP(k)=KP*eP(k)

yP(k+1)=y_coef(1)*yP(k)+y_coef(2)*uP(k)

k	r(k)	eP(k)	uP(k)	yP(k)	KP
0	200	200	160	0	0.8
1	200	124.8	99.84	75.2	



Integral (I) Control $K(z) = \frac{K_I z}{z-1}$

eI(k) = r(k)-yI(k)

uI(k)=uI(k-1)+KI*eI(k)

yI(k+1)=y_coef(1)*yI(k)+y_coef(2)*uI(k)

eI(k)	uI(k)	yI(k)	KI
200	80	0	0.4
162.4	144.96	37.6